EXERCISE - 5

Page No.:
Date: / /

Q1 Find an exponential growth model $y = y_0 e^{kt}$ that satisfies the stated conditions

(i) $y_0 = 1$ and doubling time $t = 5$

(ii) $y(0) = 5$ and growth rate $= 2\%$

(iii) $y(1) = 1$ and $y(10) = 100$

(i) Given growth model is

$$y = y_0 e^{kt}$$

when $t = 0$, $y_0 = 1$

$$y = 1 \times e^0 = 1$$

when $t = 5$, $y = 2y_0$

$$\therefore 2y_0 = y_0 e^{k5}$$

$$2 = e^{5k}$$

$$5k = \log_e 2$$

$$5k = 0.3010 \times 2.303$$

$$k = \frac{0.3010 \times 2.303}{5}$$

$$k = 0.1386$$

$$y = 1 \times e^{0.1386t}$$

$$y = e^{0.1386t}$$

ii) Equation of growth model is

$$y = y_0 e^{kt} \quad \text{---} (1)$$

$t = 0$, $y = 5$

$$\therefore 5 = y_0 e^0$$

$$5 = y_0$$

Diff wrt t (1)

$$\frac{dy}{dt} = y_0 k e^{kt}$$

Page No.:

Date: / /

at $t = 0$, $\frac{dy}{dt} = \frac{2y_0}{100}$

$$\frac{2}{100} y_0 = y_0 k e^0$$

$$k = \frac{2}{100}$$

$$y = 5e^{2/100\, t} = 5e^{0.02t}$$

iii) Equation of growth model is

$$y = y_0 e^{kt}$$

when $t = 1$, $y = 1$

$$1 = y_0 e^k \quad \text{——— (1)}$$

when $t = 10$, $y = 100$

$$100 = y_0 e^{10k} \quad \text{——— (2)}$$

Divide eq^n (2) by (1) $\quad 100 = \frac{e^{10k}}{e^k}$

$$100 = e^{9k} \Rightarrow \log_e 100 = 9k$$

$$2 \log_e 10 = 9k$$

$$2 \times 2.303 = 9k \Rightarrow \frac{2 \times 2.303}{9} = k$$

$0.511 = k$ $\quad$ Put in (1)

$$1 = y_0 e^{0.511}$$

$$y_0 = \frac{1}{e^{0.511}} = e^{-0.511}$$

$$y = e^{-0.511} e^{0.511t}$$

$$\approx 0.60\, e^{0.511t}$$

$$e^{-0.511} = \frac{1}{e^{0.511}} = \frac{1}{(2.718)^{0.511}} \approx \frac{1}{\sqrt{2.718}}$$

$$\approx \frac{1}{1.648}$$

$$\approx 60$$

Page No.:
Date: / /

Q3 In a certain culture of bacteria the number of bacteria increased 5 times in 10 hours. How long did it take for the number of bacteria to double.

Solution Let P be the number of bacteria after time t years

$\therefore \frac{dP}{dt} \propto P$

$$\frac{dP}{dt} = KP$$

$$\int \frac{dP}{P} = K \int dt$$

$$\log_e P = Kt + C$$

$$P = e^{Kt + C_1}$$

$$P = e^{Kt} e^{C} \quad (1)$$

$$P = A e^{Kt} \quad (\text{where } A = e^{C})$$

Let at $t = 0$, $P = P_1$

Put in (1)

$$P_1 = A e^{0}$$

$$A = P_1$$

$$\boxed{P = P_1 e^{Kt}}$$

ATQ at $t = 10$, $P = 5P_1$

$$\therefore 5P_1 = P_1 e^{10K}$$

$$5 = e^{10K}$$

$$10K = \log_e 5$$

$$10K = 1.61$$

$$K = 0.161$$

$$P = P_1 e^{0.161 t}$$

Let at $t = T$, $P = 2P_1$

Page No.:
Date: / /

$\therefore\ 2P_1 = P_1 e^{0.161T}$

$\log_e 2 = 0.161T$

$0.6931 = 0.161T$

$T = \dfrac{0.6931}{0.161}$

$T = 4.3\ hr$

$\therefore$ No. of bacteria will be double in 4.3 hours.

Q11 Use the exponential growth model to show that the time it takes for a population to double (i.e from an initial number A to 2A) is given by $t = \dfrac{\ln 2}{K}$.

Solution: Let P be the population at time (t).

$\therefore \dfrac{dP}{dt} \propto P$

$\dfrac{dP}{dt} = KP$

$\int \dfrac{dP}{P} = K\int dt$

$\log P = Kt + C$

$P = e^{Kt} \cdot e^{C}$

$P = \lambda e^{Kt}$ (where $e^{C} = \lambda$)

Let at $t = 0$, $P = A$ (Initially)

$\therefore A = \lambda e^{0}$

$\lambda = A$

$\therefore P = Ae^{Kt}$

Let at time t, $P = 2A$

$\therefore 2A = Ae^{Kt}$

Page No.:
Date: / /

$$2 = e^{kt}$$

$$\log_e 2 = kt$$

$$t = \frac{\log_e 2}{k}$$

Q4 The amount of oil pumped from one of the wells decreases at the continuous rate of 10% per year. When will the well's output fall to one-fourth of its present value?

Solution Let P be the oil pumped in time t

then ATQ

$$\frac{dP}{dt} = \frac{-10}{100}P$$

$$\int \frac{dP}{P} = \frac{-1}{10}\int dt$$

$$\log_e P = \frac{-1}{10}t + c$$

$$P = e^{-\frac{1}{10}t + c}$$

$$= e^{-\frac{1}{10}t} \cdot e^{c}$$

$$P = \lambda e^{-\frac{1}{10}t} \quad (\text{where } e^c = \lambda)$$

Let initially at $t = 0$, $P = P_1$

$\therefore \boxed{P_1 = \lambda}$

Hence $P = P_1 e^{-\frac{1}{10}t}$

Now let at $t = T$, $P = \frac{1}{4}P_1$

$$\therefore \frac{1}{4}P_1 = P_1 e^{-\frac{1}{10}T}$$

$$\frac{1}{4} = e^{-\frac{1}{10}T}$$

$$\log_e\left(\frac{1}{4}\right) = \frac{-1}{10}T$$

$$-\log_e 4 = \frac{-1}{10}T$$

$$2\log_e 2 = \frac{1}{10}T$$

$$2 \times 0.6931 = \frac{1}{10}T$$

$$2 \times 10 \times 0.6931 = T$$

$$13.8620 = T$$

wells output fall to one-fourth in 13.8 years.

Q8 Half-life of radioactive carbon-14 is 5700 years. A ~~ce~~ certain bone was observed to contain 75% of carbon-14 as compared to what is present in the leaving creatures. Determine its antiquity.

Solution Let (P) be the mass of carbon-14 at time (t)

$$\therefore \frac{dP}{dt} \propto (-P) \Rightarrow \frac{dP}{dt} = -kP$$

$$\int \frac{dP}{P} = -k\int dt$$

$$\log_e P = -kt + c$$

$$P = e^{-kt+c}$$

$$P = e^{-kt}e^{c}$$

Page No.:
Date: / /

$$P = \lambda e^{-kt} \quad (\text{where } e^{c} = \lambda)$$

Let at $t = 0$ (initially), $P = P_0$

$$P_0 = \lambda e^{0}$$

$$\boxed{\lambda = P_0}$$

$$P = P_0 e^{-kt}$$

at $t = 5700$ years, $P = \frac{P_0}{2}$

$$\frac{P_0}{2} = P_0 e^{-5700k}$$

$$\frac{1}{2} = e^{-5700k}$$

$$\log \frac{1}{2} = -5700k$$

$$-\log 2 = -5700k$$

$$0.6931 = 5700k$$

$$K = \cancel{5700k}\ \frac{0.6931}{5700}$$

$$P = P_0 e^{-\frac{0.6931}{5700} t}$$

Let at $t = T$, $P = \frac{75 P_0}{100}$

$$\frac{75}{100} P_0 = P_0 e^{-\frac{0.6931}{5700} T}$$

$$\frac{3}{4} = e^{-\frac{0.6931}{5700} T}$$

$$\log_e 3 - \log_e 4 = -\frac{0.6931 T}{5700}$$

Page No.:
Date: / /

$$-0.2876 = \frac{-0.6931T}{5700}$$

$$\frac{0.2876 \times 5700}{0.6931} = T$$

$$2365.19 = T$$

Antiquity of bone is 2365.19 years.
Ans

Q2 Gaurav deposited ₹ 5000 in an account paying 3% interest compounded continuously for 5 years.
(i) Find the total amount at the end of 5 years
(ii) How long will it take for the money to double?

Solution Let A be the amount at time t compounded continously at rate r%.

$$\frac{dA}{dt} = \frac{r}{100} A$$

$$\int \frac{dA}{A} = \frac{r}{100} \int dt$$

$$\log A = \frac{rt}{100} + C$$

$$A = e^{\frac{rt}{100} + C}$$

$$A = e^{\frac{rt}{100}} e^{C}$$

$$A = A_0 \, e^{rt/100} \quad \text{--- (i)}$$

at $t = 0$, $A = ₹\ 5000$

$$5000 = A_0$$

Page No.:
Date: / /

Thus $A = 5000 e^{3t/100}$ — (2)

$A = 5000 e^{0.03t}$

(i) at $t = 5$

Amount $A = 5000 e^{0.03 \times 5}$

$= 5000 e^{0.15}$

$= 5000 \times 1.1618$

$= ₹\ 5809.17$

(ii) Let at $t = T$, money will double

$\therefore$ from equation (2)

$$2A_0 = A_0 e^{0.03T}$$

$$2 = e^{0.03T}$$

$$\log_e 2 = 0.03T$$

$$\Rightarrow 0.6931 = 0.03T$$

$$T = \frac{0.6931}{0.03} = 23.10 \text{ years}$$

Q5 A cup of tea with temperature 95°C place in a room with a constant temperature of 21°C How many minutes will it take to reach a temperature of 51°C if it cools to 85°C in 1 minute

Solution Let T be the temperature of tea at time (t)

$\therefore$ By Newtons Law of cooling The rate of change of temp (T) of of body is proportional to the difference between (T) and the

Page No.:
Date: / /

temperature of surrounding medium A

$$\frac{dT}{dt} \propto (T-A)$$

$$\frac{dT}{dt} = -k(T-A)$$

$$\int \frac{dT}{T-A} = -k\int dt$$

$$\log(T-A) = -kt + c$$

$$\log(T-21) = -kt + c$$

$$T-21 = e^{-kt+c}$$

$$T-21 = e^{-kt}e^{c}$$

$$T-21 = \lambda e^{-kt} \quad \text{--- (1)}$$

where $\lambda = e^{c}$

at $t=0$, $T=95$

$\therefore$ from (1)

$$95-21 = \lambda$$

$$\lambda = 74$$

$$T-21 = 74e^{-kt} \quad \text{--- (2)}$$

when $t = 1\text{min}$, $T = 85'c$

from (2)

$$85-21 = 74e^{-k}$$

$$\frac{64}{74} = e^{-k}$$

$$e^{k} = \frac{74}{64}$$

$$k = \log\frac{74}{64}$$

$$k = 0.1452$$

Put in (2)

$$T-21 = 74e^{-0.1452t}$$

when $T=51$

$$51-21 = 74e^{-0.1452t}$$

Page No.:
Date:

$$30 = 74 e^{-0.1452t}$$
$$30 = 74 e^{-0.1452t}$$
$$\frac{30}{74} = e^{-0.1452t}$$
$$0.1452t = \log_e\left(\frac{74}{30}\right)$$
$$0.1452t = 0.90286$$
$$t = \frac{0.90286}{0.1452}$$
$$t = 6.2 \text{ min}$$

Q7 Radium decomposes at a rate proportional to the amount present. If half the original amount disappears in 1600 years, find the percentage lost in 100 years.

Solution Let (P) be the amount of Radium at time (t) years

ATQ

$$\frac{dP}{dt} \propto (-P)$$
$$\frac{dP}{dt} = -kP$$
$$\int \frac{dP}{P} = -k\int dt$$
$$\log_e P = -kt + c$$
$$P = e^{-kt+c}$$
$$P = e^{-kt} e^{c}$$
$$P = \lambda e^{-kt} \quad (\lambda = e^{c})$$

at $t = 0$, Let $P = P_0$
$$P_0 = \lambda e^{0}$$

$A = P_0$

$P = P_0 e^{-kt}$

ATQ at $t = 1600$ years

$P = \frac{P_0}{2}$

$\frac{P_0}{2} = P_0 e^{-1600k}$

$\frac{1}{2} = e^{-1600k}$

$-1600k = \log 1 - \log 2$

$\cancel{-}1600k = \cancel{-}0.6931$

$k = \cancel{.69} \ \frac{0.6931}{1600}$

$P = P_0 e^{-\frac{0.6931t}{1600}}$

at $t = 100$

$P = P_0 e^{-\frac{0.6931 \times 100}{1600}}$

$\frac{P}{P_0} = e^{-\frac{0.6931}{16}}$

$\frac{P_0}{P} = e^{\frac{0.6931}{16}}$

$\log_e \left(\frac{P_0}{P} \right) = \frac{0.6931}{16}$

$\log_e \left(\frac{P_0}{P} \right) = 0.04331$

$\frac{P_0}{P} = e^{0.04331}$

$\frac{P_0}{P} = 1.0442$

$\frac{P}{P_0} = \frac{1}{1.0442}$

Page No.:
Date: / /

$$\frac{P}{P_0} = 0\cdot9577$$

$$P = 0\cdot9577 P_0$$

Radium lost in 100 years

$$= P_0 - 0\cdot9577 P_0$$

$$= 0\cdot0423 P_0$$

$$= \frac{0\cdot0423 P_0}{P_0} \times 100\%$$

$$= 4\cdot2\%$$